International Journal of Solids and Structures 47 (2010) 3297–3304

Contents lists available at ScienceDirect

International Journal of Solids and Structures

journal homepage: www.elsevier.com/locate/ijsolstr

Plane elasticity problem for a multi-wedge system with a thin wedge

Alexander Linkov [a,1], Liliana Rybarska-Rusinek [b,*]

[a] Institute for Problems of Mechanical Engineering, Russian Academy of Sciences, 61 Bol'shoy pr. V.O., St. Petersburg 199178, Russia
[b] The Faculty of Mathematics and Applied Physics, Rzeszow University of Technology, W. Pola 2, 35-959 Rzeszow, Poland

ARTICLE INFO

Article history:
Received 11 February 2010
Received in revised form 9 July 2010
Available online 20 August 2010

Keywords:
Plane elasticity
Multi-wedge points
Asymptotics
Interaction models

ABSTRACT

The paper presents a method for studying a system of elastic wedges containing a thin wedge with the angle Θ_0, which may be arbitrary small. An analysis shows that the considered problem, involving 2-D vectors of tractions and displacements, cannot be solved by straight-forward extension of the method previously worked out by the authors for analogous scalar problems. The difficulty arises because of the disclosed feature of the dependences between the Mellin transformed displacements and tractions at the boundaries of a thin wedge: they are linearly dependent when their Taylor's expansions in Θ_0 are represented by the first terms only. The difficulty is removed by using the consequences of the linear dependence and by an appropriate re-arrangement of variables. Then simple physical models, simulating the influence of a thin wedge on a multi-wedge system, become available. The models cover the cases of a very rigid and very compliant thin wedge and also intermediate cases. The ranges of the models applicability are studied analytically and illustrated by numerical results.

1. Introduction

The need in studying multi-wedge systems arises in solid mechanics because of growing interest in accounting for an internal structure of natural and artificial materials and also because of the strong influence of common apexes of wedges on the accuracy of numerical calculations (see, e.g., the reviews in Blinova and Linkov (1995), Linkov and Koshelev (2005), Paggi and Carpinteri (2008), Sinclair (2004a,b)). The singular behavior of physical fields in the neighborhood of a common apex of wedges strongly depends on the contact conditions at wedge interfaces. The influence of contact conditions appears quite obviously in the simplest case of a frictionless contact. It was firstly studied by Dundurs and Lee (1972) for a wedge contacting with a half-plane. The authors reported that in this case (i) there was no complex roots of the characteristic determinant, and (ii) the singularity was not stronger than the inverse root of the distance d to the wedge apex. No complex roots were reported later for the Coulomb law of dry friction by Gdoutos and Theocaris (1975), Comninou (1976) and Churchman et al. (2004), and for an arbitrary system with frictionless contacts by Linkov and Rybarska-Rusinek (2010). Meanwhile, the extended studying has shown that in some cases the power of the singularity may be stronger than $O(1/\sqrt{d})$ both for dry friction (Churchman et al., 2004) and for smooth contacts (Linkov and Rybarska-Rusinek, 2010). For the Coulomb friction law, it depends on the slip direction what is of importance for fretting problems (Churchman et al., 2004; Mugadu et al., 2004).

From the physical point of view, the contact conditions reflect processes in a thin layer with properties, which may be quite different from those of embedding wedges. Hence, to properly model contact interaction near singular points, it is reasonable to study systems including a wedge with a small angle Θ_0, which in limit may turn to zero. The first paper, tending to derive models of contact interaction by studying an exact solution for a thin wedge (Mishuris and Kuhn, 2001), considered a particular case of anti-plane-strain problem for a semi-infinite crack on the boundary of dissimilar materials with a thin wedge ahead of the crack tip. The authors used the Mellin transform and derived two models of contact interaction, which compliment each other in the cases of a very compliant and very rigid thin wedge. The extension of these results to an arbitrary system of wedges under anti-plane-strain conditions is given in Linkov and Rybarska-Rusinek (2008) by separating the symmetric and anti-symmetric parts of the solution and by representing trigonometric multipliers by truncated Taylor's series. This approach provides a general model of contact interaction, which includes those of the paper (Mishuris and Kuhn, 2001) as particular cases. A further step in developing a general technique for simulating the influence of a thin wedge consisted in distinguishing those quantities, which do not turn to infinity when the angle Θ_0 tends to zero, and in solving the equations for embedding wedges with respect to these quantities (Linkov and Rybarska-Rusinek, 2010). The approach proved to provide simple physical models and remarkably accurate numerical results

* Corresponding author.
E-mail address: rybarska@prz.edu.pl (L. Rybarska-Rusinek).
[1] Present address: The Faculty of Mathematics and Applied Physics, Rzeszow University of Technology, W. Pola 2, 35-959 Rzeszow, Poland.

0020-7683/$ - see front matter © 2010 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ijsolstr.2010.08.004

(Linkov and Rybarska-Rusinek, 2010) in the particular case of smooth contacts in the plane-strain or plane-stress problems. In this case, the problem reduces to equations for *scalar* quantities.

It seemed that the approach could be promptly extended to *vector* cases of plane-strain (plane-stress) problems. However, it has appeared that it was not true because of an unexpected feature of the truncated equations in these cases: the equations become linearly dependent when neglecting quite small terms of order Θ_0^3 as compared with the unit. Below we overcome the difficulty by using the very reason, which causes it (linear dependence), by an appropriate re-arrangement of variables and by comparing the terms entering the resulting equations. This serves us to obtain simple physical models, simulating the influence of a thin wedge on a multi-wedge system, and to study the ranges of their applicability. The models cover the cases of a very rigid and very compliant thin wedge and also intermediate cases. Numerical results illustrate the accuracy provided by each of the models within the range of its applicability.

2. Problem formulation: Numerical solution of Mellin transformed equations

Consider a system open (Fig. 1a) or closed (Fig. 1b) of an arbitrary number of elastic wedges under the plane-strain or plane-stress conditions. For an open system, we assume prescribed tractions $\sigma_{\vartheta\vartheta}(r)$, $\sigma_{r\vartheta}(r)$ or prescribed displacements $u_\vartheta(r)$, $u_r(r)$ at the outer boundaries ((r,ϑ) are polar coordinates with the center at the common apex of the wedges). At contacts of wedges, there might be prescribed displacement discontinuities $\Delta u_\vartheta(r)$, $\Delta u_r(r)$ or/and traction discontinuities $\Delta\sigma_{\vartheta\vartheta}(r)$, $\Delta\sigma_{r\vartheta}(r)$. They may arise because of such reasons as thermal or poro-elastic effects (e.g. (Mishuris, 1997; Dobroskok and Linkov, 2010)). Their explicit forms are obtained, for instance, by using superposition of particular solutions for the temperature (pore pressure) and additional fields satisfying homogeneous PDE.

Indeed, in thermo- (poro-) elastic problems, a displacement field satisfies inhomogeneous PDE, which besides the Lame's operator, applied to displacements, includes also gradient of temperature (pore pressure). Therefore, to apply a method developed for homogeneous PDE, the displacements u_i, stresses σ_{ij} and tractions σ_{ni} are represented as the sums:

$$u_i = u_i^p + u_i', \quad \sigma_{ij} = \sigma_{ij}^p + \sigma_{ij}', \quad \sigma_{ni} = \sigma_{ni}^p + \sigma_{ni}',$$

where the parts with the superscript p represent arbitrary particular solutions of the inhomogeneous PDE, while the parts with prime, called additional, satisfy the common homogeneous PDE of the elasticity theory. A problem is reformulated in terms of the additional quantities, for which a common way of analysis becomes available. However, in contrast with the displacements u_i and tractions σ_{ni}, the additional quantities are commonly discontinuous at contacts of regions, which have different elastic and/or thermal (porous) properties, because, as a rule, particular solutions are discontinuous at the contacts. In this paper, we assume a problem formulated in terms of the additional quantities to use the analysis applicable to the homogeneous PDE. The prime in the notation of additional quantities is omitted. Normally, particular displacements are smooth functions of the coordinate tangential to a contact; consequently, the discontinuities arising in the additional displacements and tractions are also smooth functions of this coordinate. Thus in the further discussion, it may be assumed that the discontinuities along a contact Δu_ϑ, Δu_r, $\Delta\sigma_{\vartheta\vartheta}$, $\Delta\sigma_{r\vartheta}$ are sufficiently smooth functions of the polar coordinate r near a common apex. For this reason, below starting from Eq. (4) they are set zero. Final results do not depend on them.

The system contains a thin wedge with the angle Θ_0, shear modulus μ_0 and Poisson's ratio ν_0. The angle Θ_0 may be arbitrary small, in limit zero. Our aim is to find models, which simulate the influence of the thin wedge by dependences between the displacements and tractions at the surfaces of wedges embedding the thin wedge.

The notation, starting equations and method for their efficient and accurate solving are those of the papers (Blinova and Linkov, 1995; Linkov and Koshelev, 2005). We use the Mellin transformed tractions, displacements, PDE, boundary and contact conditions. The transformation parameter s, written as an argument of a quantity, signifies its Mellin transformed value; the argument r refers to a physical quantity. For the further discussion, it is sufficient to know that for any value of the transformation parameter s, the contact values of transformed tractions and displacements are easily found by applying Gauss pivotal elimination to the system of three-point difference equations derived in Blinova and Linkov (1995). All the formulae used to make this procedure efficient, stable and accurate are written explicitly in Linkov and Koshelev (2005)). In the present paper we use a computer code based on them and assume that a solution is known for any multi-wedge system, which does not contain a wedge, whose angle may be arbitrary small, in limit zero. Meanwhile, if a system includes a thin wedge with a finite although very small angle of some minutes, the code, in accordance with the results of Linkov and Koshelev (2005), still gives us accurate values of all the coefficients of the algebraic system, and also its determinant and its roots under prescribed boundary and contact conditions specified above. This serves us for additional control of calculations performed by methods discussed below. Having this in mind we may focus on accounting for a thin wedge with the angle which may be arbitrary small, in limit zero.

3. Solutions for external system and thin wedge

First of all, we need to distinguish the influence of a thin wedge. To this end, we consider separately the thin wedge (Fig. 2a) and the system of remaining wedges (Fig. 2b), external to it (for an open

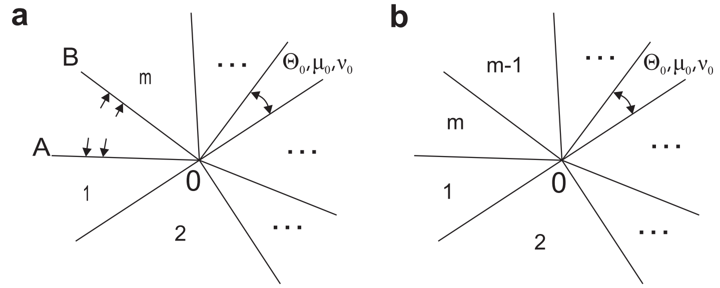

Fig. 1. A system of m wedges a) open, b) closed.

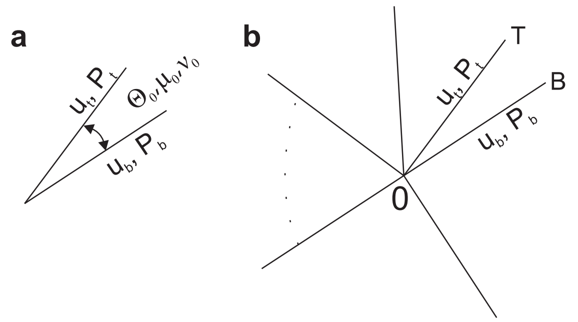

Fig. 2. Schemes of a) a thin wedge, b) a system open or closed without the thin wedge.

system, there are two external systems). All the quantities used in this section refer to their Mellin transformed values.

3.1. External system

For an external system, the accurate solution is found by the Gauss pivotal elimination used in a code developed. To match the external system with the thin wedge, we firstly find the displacements at the boundaries *OT* and *OB* for four sets of boundary conditions under homogeneous conditions on contacts: 1) $p_{t\vartheta}=1$, $p_{tr}=p_{b\vartheta}=p_{br}=0$, 2) $p_{tr}=1$, $p_{t\vartheta}=p_{b\vartheta}=p_{br}=0$, 3) $p_{b\vartheta}=1$, $p_{t\vartheta}=p_{tr}=p_{br}=0$, 4) $p_{br}=1$, $p_{t\vartheta}=p_{tr}=p_{b\vartheta}=0$, where the subscript t refers to the boundary *OT*, the subscript b to the boundary *OB*. Denote $d_{ij}(s)$ the displacements $u_{t\vartheta}$, u_{tr}, $u_{b\vartheta}$ and u_{br} for $i=1,2,3$ and 4, respectively, obtained from the solution of the problem j ($j=1,2,3,4$). Secondly, by the same code, we may find the solution induced by prescribed displacement and traction discontinuities at contacts under zero tractions at the boundaries *OT* and *OB*. Denote $v_{t\vartheta}$, v_{tr}, $v_{b\vartheta}$and v_{br} the displacements obtained at the boundaries *OT* and *OB*. Then by superposition, for arbitrary tractions $p_{t\vartheta}$, p_{tr} on the boundary *OT* and arbitrary tractions $p_{b\vartheta}$, p_{br} on the boundary *OB*, the corresponding displacements $u_{t\vartheta}$, u_{tr}, $u_{b\vartheta}$ and u_{br} on these boundaries are given by

$$U_e(s)=D(s)P_e(s)+V_e(s), \tag{1}$$

where $U_e(s)$ and $P_e(s)$ are vectors of external displacements and tractions, respectively, $V_e(s)$ is a known vector corresponding to the prescribed discontinuities on contacts:

$$U_e(s)=\begin{pmatrix}u_t\\u_b\end{pmatrix},\quad P_e(s)=\begin{pmatrix}p_t\\p_b\end{pmatrix},\quad V_e(s)=\begin{pmatrix}v_t\\v_b\end{pmatrix};$$
$$u_t=\begin{pmatrix}u_{t\vartheta}\\u_{tr}\end{pmatrix},\quad u_b=\begin{pmatrix}u_{b\vartheta}\\u_{br}\end{pmatrix},\quad p_t=\begin{pmatrix}p_{t\vartheta}\\p_{tr}\end{pmatrix},\quad p_b=\begin{pmatrix}p_{b\vartheta}\\p_{br}\end{pmatrix},$$
$$v_t=\begin{pmatrix}v_{t\vartheta}\\v_{tr}\end{pmatrix},\quad v_b=\begin{pmatrix}v_{b\vartheta}\\v_{br}\end{pmatrix}.$$

$D(s)$ is a 4×4 matrix with the components $d_{ij}(s)$.

3.2. Thin wedge

For a thin wedge, the general equations, connecting the boundary values of the displacements and tractions on its boundaries (Blinova and Linkov, 1995; Linkov and Koshelev, 2005), are of use:

$$U_0(s)=R_0(s)P_0(s), \tag{2}$$

where $U_0(s)$ and $P_0(s)$ are vectors of displacements and tractions at the boundaries of the thin wedge:

$$U_0(s)=\begin{pmatrix}u_{0t}\\u_{0b}\end{pmatrix},\quad P_0(s)=\begin{pmatrix}p_{0t}\\p_{0b}\end{pmatrix};$$
$$u_{0t}=\begin{pmatrix}u_{0t\vartheta}\\u_{0tr}\end{pmatrix},\quad u_{0b}=\begin{pmatrix}u_{0b\vartheta}\\u_{0br}\end{pmatrix},\quad p_{0t}=\begin{pmatrix}p_{0t\vartheta}\\p_{0tr}\end{pmatrix},\quad p_{0b}=\begin{pmatrix}p_{0b\vartheta}\\p_{0br}\end{pmatrix},$$
$$R_0(s)=\frac{1}{2\mu_0(s+1)}\frac{1}{T_0(s)}\begin{pmatrix}R_{0tt}&R_{0tb}\\R_{0bt}&R_{0bb}\end{pmatrix},$$
$$T_0(s)=T_0^S(s)T_0^A(s),$$
$$T_0^S(s)=(s+1)\sin\Theta_0+\sin(s+1)\Theta_0,$$
$$T_0^A(s)=(s+1)\sin\Theta_0-\sin(s+1)\Theta_0,$$
$$R_{0tt}(s)=\frac{1}{2}(T_0^AR_0^S+T_0^SR_0^A),\quad R_{0tb}(s)=-\frac{1}{2}(T_0^AR_0^S-T_0^SR_0^A)G,$$
$$R_{0bt}(s)=\frac{1}{2}G(T_0^AR_0^S-T_0^SR_0^A),\quad R_{0bb}(s)=-\frac{1}{2}G(T_0^AR_0^S+T_0^SR_0^A)G,$$
$$R_0^S(s)=\begin{pmatrix}k_0a_{0-}&-T_0^S+k_0b_{0+}\\T_0^S+k_0b_{0-}&-k_0a_{0+}\end{pmatrix},$$
$$R_0^A(s)=\begin{pmatrix}k_0a_{0+}&-T_0^A+k_0b_{0-}\\T_0^A+k_0b_{0+}&-k_0a_{0-}\end{pmatrix},$$
$$a_{0\pm}=2(\cos\Theta_0\pm\cos(s+1)\Theta_0),\quad b_{0\pm}=2(\sin\Theta_0\pm\sin(s+1)\Theta_0),$$

$k_0=1-v_0$ for plane-strain, $k_0=\frac{1}{1+v_0}$ for plane-stress, the subscript 0 indicates that a value refers to the thin wedge. G is the constant matrix serving to distinguish the symmetric and anti-symmetric parts of a solution:

$$G=\begin{pmatrix}-1&0\\0&1\end{pmatrix}.$$

3.3. Equations for the joint external system and thin wedge

In a general case, there might be prescribed discontinuities of the displacements and tractions

$$\Delta v_{0t}=\begin{pmatrix}\Delta v_{0t\vartheta}\\\Delta v_{0tr}\end{pmatrix},\quad \Delta v_{0b}=\begin{pmatrix}\Delta v_{0b\vartheta}\\\Delta v_{0br}\end{pmatrix},\quad \Delta p_{0t}=\begin{pmatrix}\Delta p_{0t\vartheta}\\\Delta p_{0tr}\end{pmatrix},$$
$$\Delta p_{0b}=\begin{pmatrix}\Delta p_{0b\vartheta}\\\Delta p_{0br}\end{pmatrix},$$

at the contacts between the external system and the thin wedge. Denote $\Delta V_0=U_e-U_0$, $\Delta P_0=P_e-P_0$ the corresponding vectors of generalized prescribed discontinuities:

$$\Delta V_0(s)=\begin{pmatrix}\Delta v_{0t}\\\Delta v_{0b}\end{pmatrix},\quad \Delta P_0(s)=\begin{pmatrix}\Delta p_{0t}\\\Delta p_{0b}\end{pmatrix}.$$

Then (1) and (2) yield

$$DP_0=R_0P_0+\Delta V, \tag{3}$$

where $\Delta V=\Delta V_0+V_e-D\Delta P_0$ is a known vector induced by the prescribed discontinuities at the interfaces of the joint system.

For further discussion, it is convenient to represent the values on contacts *OT* and *OB* by the sums of their symmetric and anti-symmetric parts, having clear physical interpretation,

$$U_e=H_U\begin{pmatrix}U^S\\U^A\end{pmatrix},\quad V_e=H_U\begin{pmatrix}V^S\\V^A\end{pmatrix},\quad U_0=H_U\begin{pmatrix}U_0^S\\U_0^A\end{pmatrix},$$
$$\Delta V_0=H_U\begin{pmatrix}\Delta V_0^S\\\Delta V_0^A\end{pmatrix},\quad \Delta V=H_U\begin{pmatrix}\Delta V^S\\\Delta V^A\end{pmatrix},$$
$$P_e=H_P\begin{pmatrix}P^S\\P^A\end{pmatrix},\quad P_0=H_P\begin{pmatrix}P_0^S\\P_0^A\end{pmatrix},\quad \Delta P_0=H_P\begin{pmatrix}\Delta P_0^S\\\Delta P_0^A\end{pmatrix},$$

where the symmetric U^S, P^S and anti-symmetric U^A, P^A parts of U_e are 2-D vectors defined as:

$$U^S=\begin{pmatrix}u_\vartheta^S\\u_r^S\end{pmatrix}=\frac{1}{2}(u_t+Gu_b),\quad U^A=\begin{pmatrix}u_\vartheta^A\\u_r^A\end{pmatrix}=\frac{1}{2}(u_t-Gu_b),$$
$$P^S=\begin{pmatrix}p_\vartheta^S\\p_r^S\end{pmatrix}=\frac{1}{2}(p_t-Gp_b),\quad P^A=\begin{pmatrix}p_\vartheta^A\\p_r^A\end{pmatrix}=\frac{1}{2}(p_t+Gp_b);$$
$$H_U=\begin{pmatrix}I&I\\G&-G\end{pmatrix},\quad H_P=\begin{pmatrix}I&I\\-G&G\end{pmatrix},$$

the constant matrix G is defined above. The definitions of other vectors are similar with obvious changes in symbols of vectors and indices. Note that the matrices H_U and H_P are not singular and have the same determinant: $\det H_U=\det H_P=-4$. Their inverses are:

$$H_U^{-1}=\frac{1}{2}\begin{pmatrix}I&G\\I&-G\end{pmatrix},\quad H_P^{-1}=\frac{1}{2}\begin{pmatrix}I&-G\\I&G\end{pmatrix}.$$

When looking for models simulating the influence of a thin wedge, we are interested in the asymptotic behavior of the solution defined by the properties of the thin wedge itself. Consequently, we do not discuss singular behavior, which may be induced by particular singular external loads. For this reason, we have assumed that the prescribed physical discontinuities, resulting from external factors, are sufficiently smooth functions. Actually, it is sufficient to assume that they do not have a singularity stronger than that induced by the presence of the thin wedge. This implies that only the roots of the determinant $\det(D-R_0)$ are of interest for our purpose. Henceforth, we focus on the homogeneous versions of the previous equations and set $V_e(s)=0$, $\Delta V_0(s)=0$, $\Delta P_0(s)=0$. Then $\Delta V=0$, and Eq. (3) becomes

$$DP_0=R_0P_0. \tag{4}$$

In terms of the symmetric and anti-symmetric parts, Eqs. (1) and (4) become, respectively,

$$\begin{pmatrix}U^S\\U^A\end{pmatrix}=A(s)\begin{pmatrix}P^S\\P^A\end{pmatrix}, \tag{5}$$
$$\begin{pmatrix}U_0^S\\U_0^A\end{pmatrix}=A_0(s)\begin{pmatrix}P_0^S\\P_0^A\end{pmatrix}, \tag{6}$$

where

$$A(s)=H_U^{-1}D(s)H_P,\quad A_0(s)=H_U^{-1}R_0(s)H_P.$$

In accordance with the said above, we have neglected the free term in (1). The equation for the joined system (3) reads

$$A(s)\begin{pmatrix}P_0^S\\P_0^A\end{pmatrix}=A_0(s)\begin{pmatrix}P_0^S\\P_0^A\end{pmatrix}. \tag{7}$$

Note that, since $\det H_U^{-1}=1/\det H_P$, we have $\det(A-A_0)=\det(D-R_0)=\Delta(s)$. Therefore, the needed roots of the determinant $\Delta(s)$ in the strip $-2<s<-1$ may be found from the equation

$$\det(A-A_0)=0. \tag{8}$$

Below we shall use also the inversions of (5)–(7). They are

$$\begin{pmatrix}P^S\\P^A\end{pmatrix}=B(s)\begin{pmatrix}U^S\\U^A\end{pmatrix}, \tag{9}$$

$$\begin{pmatrix}P_0^S\\P_0^A\end{pmatrix}=B_0(s)\begin{pmatrix}U_0^S\\U_0^A\end{pmatrix}, \tag{10}$$

$$B(s)\begin{pmatrix}U_0^S\\U_0^A\end{pmatrix}=B_0(s)\begin{pmatrix}U_0^S\\U_0^A\end{pmatrix}. \tag{11}$$

Thus, alternatively, we may find the needed roots from the equation

$$\det(B-B_0)=0. \tag{12}$$

Note that the last equation is actually obtained by expressing all the tractions in (1) via all the displacements and applying a similar operation to (2). In similar way, we may obtain other equations equivalent to (7) by solving both (1) and (2) with respect to other sets of four variables. Below we shall use this option to obtain simple models, which simulate the influence of a thin wedge.

4. Approximate equations for truncated Taylor's expansions

In the explicit form the Eq. (6) reads

$$\begin{aligned}
u_\vartheta^S&=\frac{1}{2\mu_0(s+1)T^S(s)}\left[k_0a_-p_\vartheta^S+(-T^S+k_0b_+)p_r^S\right],\\
u_r^S&=\frac{1}{2\mu_0(s+1)T^S(s)}\left[(T^S+k_0b_-)p_\vartheta^S-k_0a_+p_r^S\right],\\
u_\vartheta^A&=\frac{1}{2\mu_0(s+1)T^A(s)}\left[k_0a_+p_\vartheta^A+(-T^A+k_0b_-)p_r^A\right],\\
u_r^A&=\frac{1}{2\mu_0(s+1)T^A(s)}\left[(T^A+k_0b_+)p_\vartheta^A-k_0a_-p_r^A\right].
\end{aligned} \tag{13}$$

As could be expected, in accordance with their physical meaning, the symmetric and anti-symmetric parts in (13)) are separated. This makes them convenient for further analysis. We see that the coefficients of the matrix A_0 in (6) depend on trigonometric functions of s entering expressions for entries of R_{0tt}, R_{0tb}, R_{0bt}, R_{0bb}. Although holomorphic, they do not have analytical originals in the physical plane. This prevents obtaining analytical relations between the boundary values of physical displacements and tractions by inverting (13). The same refers also to Eq. (10). Note, however, that, as clear from (13) and expressions for its entries given above, the parameter s enters the arguments of trigonometric functions only in the form of the products $(s+j)\Theta_0$ with integer j (positive, negative or zero). Then by expanding the coefficients in Taylor's series in Θ_0 we obtain them as series in s. Each of their terms has an analytical inversion, so that truncated series provide analytical expressions between displacements and tractions on the boundaries of a thin wedge; the inverted terms contain growing degrees Θ_0^k of Θ_0 $(k=1,2,\ldots)$. Naturally, to obtain the simplest models, it is reasonable to truncate the series with a minimal number of terms. For a sufficiently small angle Θ_0, only these terms actually define the physical response of the wedge.

The approximate form of (6), truncated in this way, reads

$$\begin{pmatrix}U^S\\U^A\end{pmatrix}=A_{0apx}(s)\begin{pmatrix}P^S\\P^A\end{pmatrix}. \tag{14}$$

When writing (14) explicitly, we arrive at the equations

$$\begin{aligned}
u_\vartheta^S&=\frac{1}{2\mu_0(s+1)^2}\left\{\frac{1}{2}k_0s(s+2)\Theta_0p_\vartheta^S+[(k_0-1)s+(2k_0-1)]p_r^S+O(\Theta_0^3)\right\},\\
u_r^S&=\frac{1}{2\mu_0(s+1)^2\Theta_0}\left\{[(1-k_0)s+1]\Theta_0p_\vartheta^S-2k_0p_r^S+O(\Theta_0^3)\right\},\\
u_\vartheta^A&=\frac{12}{2\mu_0(s+1)^2s(s+2)\Theta_0^3}\left\{2k_0p_\vartheta^A-k_0s\Theta_0p_r^A+O(\Theta_0^3)\right\},\\
u_r^A&=\frac{6}{2\mu_0(s+1)^2s\Theta_0^2}\left\{2k_0p_\vartheta^A-k_0s\Theta_0p_r^A+O(\Theta_0^3)\right\},
\end{aligned} \tag{15}$$

where, when evaluating the order $O(\Theta_0^3)$ of neglected terms, we have taken into account that $p_\vartheta^S(s)$, $p_r^A(s)$ are even, while $p_r^S(s)$, $p_\vartheta^A(s)$ are odd functions of Θ_0.

The Eqs. (15) have a remarkable and unexpected feature: their last two lines are *linearly dependent*. To the mentioned accuracy they imply

$$u_r^A(s) = \frac{s+2}{2}\Theta_0 u_\vartheta^A(s) + O(\Theta_0^3). \tag{16}$$

The linear dependence yields an important restriction on the applicability of the approximation (14). It can be seen from the following argument. Denote μ^* a typical shear modulus of wedges adjusting the thin wedge with the modulus μ_0. Then typical coefficients of the matrix A in (5) are of order $\frac{1}{\mu_*}O(1)$, while the r.h.s. of (14) is proportional to $1/\mu_0$. Therefore, after substitution of the approximation (14) into (7), we obtain a system, in which two last lines, corresponding to u_ϑ^A and u_r^A become practically proportional, when $\frac{\mu_*}{\mu_0\Theta_0^2} \gg 1$. Consequently, the determinant becomes practically zero for any s. This means that the approximations for anti-symmetric parts in (15) are inacceptable when a thin wedge is very compliant as compared with its neighbors, specifically, when $\frac{\mu_0}{\mu_*} \ll 1$. Obviously, that for a small angle Θ_0 this yields even stronger inequality $\frac{\mu_0}{\mu_*}\Theta_0^2 \ll 1$.

In the case $\frac{\mu_0}{\mu_*} \ll 1$, we have, as mentioned, $\frac{\mu_0}{\mu_*}\Theta_0^2 \ll 1$, and it is reasonable to start from the inverted form of (6) given by (10). Then we use the truncated expansions of the r.h.s. of (10)

$$\begin{pmatrix} P^S \\ P^A \end{pmatrix} = B_{0apx}(s)\begin{pmatrix} U^S \\ U^A \end{pmatrix}. \tag{17}$$

Written explicitly this equation reads

$$\begin{aligned}
p_\vartheta^S(s) &= \frac{2\mu_0}{(1-2k_0)\Theta_0}\{-2k_0 u_\vartheta^S(s) - [k_0(s+2)-(s+1)] \\
&\quad \times \Theta_0 u_r^S(s) + O\left(\Theta_0^3\right)\}, \\
p_r^S(s) &= \frac{2\mu_0}{1-2k_0}\left\{[k_0 s-(s+1)]u_\vartheta^S(s) + \frac{1}{2}k_0 s(s+2)\Theta_0 u_r^S(s) + O\left(\Theta_0^3\right)\right\}, \\
p_\vartheta^A(s) &= \frac{\mu_0 s}{2}\left[-(s+2)\Theta_0 u_\vartheta^A(s) + 2u_r^A(s) + O\left(\Theta_0^3\right)\right], \\
p_r^A(s) &= \frac{\mu_0}{\Theta_0}\left[-(s+2)\Theta_0 u_\vartheta^A(s) + 2u_r^A(s) + O\left(\Theta_0^3\right)\right].
\end{aligned} \tag{18}$$

where, when evaluating the order $O(\Theta_0^3)$ of neglected terms, we have also taken into account that $u_r^S(s)$, $u_\vartheta^A(s)$ are even, while $u_\vartheta^S(s)$, $u_r^A(s)$ are odd functions of Θ_0.

As could be expected, the first two equations of (18) present the inversion of the first two equations of (15). Also as expected, the last two equations in (18) are *linearly dependent*. Obviously, they do not follow from the last two equations of (15) (otherwise, their inversion would result in linearly independent expressions for $u_\vartheta^A(s)$ and $u_r^A(s)$ given by (13)). To the accuracy of terms $O(\Theta_0^3)$ the last two lines of (18) imply

$$p_\vartheta^A(s) = \frac{s}{2}\Theta_0 p_r^A(s) + O\left(\Theta_0^3\right). \tag{19}$$

Similar to the analysis for (14), it can be seen that the linear dependence yields a restriction on the applicability of the approximation (18). It appears that the approximations for anti-symmetric parts in (18) become inapplicable when a thin wedge is very rigid, specifically, when $\frac{\mu_0}{\mu_*}\Theta_0 \gg 1$. Obviously, this is the case only when $\frac{\mu_0}{\mu_*} \gg 1$.

Meanwhile, from the said it follows that the linear dependence (16) is of use for all sufficiently small Θ_0^2 including those for which $\frac{\mu_0}{\mu_*}\Theta_0^2$ is of order or not too much less than 1. Similarly, the linear dependence (19) is applicable when $\frac{\mu_0}{\mu_*}$ is of order or not too much greater than 1. Therefore, the linear dependence of the truncated anti-symmetric parts, despite generating a difficulty, gives a key for finding models simulating the influence of a thin wedge for intermediate ratios $\frac{\mu_0}{\mu_*}$. This opportunity is employed in the next section. Emphasize that the symmetric parts of (15) and (18) are applicable for a thin wedge with an arbitrary rigidity.

5. Models for thin wedge

Consider firstly the cases of a very compliant and very rigid thin wedge as compared with its neighbors.

Very compliant thin wedge $\left(\frac{\mu_0}{\mu_*} \ll 1\right)$. In this case, the inspection of the second and third lines in (9) and (18) yields equations

$$p_r^S = 0, \quad p_\vartheta^A = 0, \tag{20}$$

which mean that the *tractions are continuous* across the thin compliant wedge. Two remaining equations in (24), after the Mellin's inversion, give two additional physical dependences

$$\begin{aligned}
p_\vartheta^S(r) &= p_\vartheta(r) = \frac{2\mu_0}{1-2k_0}\left[-\frac{2k_0}{\Theta_0}u_\vartheta^S(r) + k_0 r\frac{d}{dr}\left(\frac{u_r^S}{r}\right) - \frac{du_r^S}{dr}\right], \\
p_r^S(r) &= p_r(r) = \mu_0\left[r\frac{d}{dr}\left(\frac{u_\vartheta^A}{r}\right) + \frac{2}{\Theta_0 r}u_r^A(r)\right].
\end{aligned} \tag{21}$$

Four Eqs. (20) and (21) present a physical model, which is certainly applicable for a very compliant wedge. We shall call it the *model 1*. Below it will be shown that its applicability is notably wider than that defined by the inequality $\frac{\mu_0}{\mu_*} \ll 1$.

The approximate system, corresponding to the model 1, is

$$\begin{aligned}
&\Theta_0\left(b_{11}u_\vartheta^S + b_{12}u_r^S + b_{13}u_\vartheta^A + b_{14}u_r^A\right) = \frac{2\mu_0}{1-2k_0}\{-2k_0 u_\vartheta^S(s) \\
&\qquad -\Theta_0[k_0(s+2)-(s+1)]u_r^S(s)\}, \\
&b_{21}u_\vartheta^S + b_{22}u_r^S + b_{23}u_\vartheta^A + b_{24}u_r^A = 0, \\
&b_{31}u_\vartheta^S + b_{32}u_r^S + b_{33}u_\vartheta^A + b_{34}u_r^A = 0, \\
&\Theta_0\left(b_{41}u_\vartheta^S + b_{42}u_r^S + b_{43}u_\vartheta^A + b_{44}u_r^A\right) = \mu^0\left[-(s+2)\Theta_0 u_\vartheta^A(s) + 2u_r^A(s)\right].
\end{aligned} \tag{22}$$

Note that we could neglect the terms with u_r^S and u_ϑ^A in the r.h.s. of Eqs. (9) and (18). We shall not do it to make the model applicable in the case of a moderately rigid thin wedge.

Very rigid thin wedge $\left(\frac{\mu_*}{\mu_0}\frac{1}{\Theta_0^2} \ll 1\right)$. In this case, the inspection of the first and last lines in (5) and (15) yields equations

$$u_\vartheta^S = 0, \quad u_r^A = 0, \tag{23}$$

which mean that the *displacements are continuous* across the thin rigid wedge. Two remaining equations in (15), after the Mellin's inversion, give two additional physical dependences

$$\begin{aligned}
&\frac{d}{dr}\left(r\frac{du_r}{dr}\right) = -\frac{k_0-1}{2\mu_0}\frac{d}{dr}(rp_\vartheta^S) + \frac{k_0}{2\mu^0}p_\vartheta^S(r) - \frac{k_0}{\mu^0\Theta_0}p_r^S(r), \\
&J_\vartheta\frac{1}{r^2}\frac{d^3}{dr^3}\left(r^2\frac{du_\vartheta}{dr}\right) = 2p_\vartheta^A(r) + \frac{\Theta_0}{r}\frac{d}{dr}[r^2 p_r^A(r)],
\end{aligned} \tag{24}$$

where

$$J_\vartheta = \frac{2\mu_0}{k_0}\frac{h^3}{12}, \quad h = r\Theta_0.$$

Four Eqs. (23), (24) present a physical model, which is certainly applicable for a very rigid wedge. We shall call it the *model 2*. Below it will be shown that its applicability is notably wider than that defined by the inequality $\frac{\mu_*}{\mu_0}\frac{1}{\Theta_0^2} \ll 1$.

The approximate system, corresponding to the model 2, is

$$\begin{aligned}
&a_{11}p_\vartheta^S + a_{12}p_r^S + a_{13}p_\vartheta^A + a_{14}p_r^A = 0, \\
&\Theta_0\left(a_{21}p_\vartheta^S + a_{22}p_r^S + a_{23}p_\vartheta^A + a_{24}p_r^A\right) \\
&\quad = \frac{1}{2\mu_0(s+1)^2}\{-[(k_0-1)(s+1)-k_0]\Theta_0 p_\vartheta^S - 2k_0 p_r^S\}, \\
&\Theta_0^3\left(a_{31}p_\vartheta^S + a_{32}p_r^S + a_{33}p_\vartheta^A + a_{34}p_r^A\right) \\
&\quad = \frac{12k_0}{2\mu_0(s+1)^2 s(s+2)}(2p_\vartheta^A - s\Theta_0 p_r^A),
\end{aligned}$$

$$a_{41}p_\vartheta^S + a_{42}p_r^S + a_{43}p_\vartheta^A + a_{44}p_r^A = 0. \tag{25}$$

Note that, as clear from (15), in the case considered we could set $u_r^S = 0$, as well. We could also neglect terms with $p_\vartheta^S,\ p_r^A$ in the r.h.s. of (24) and (25). We shall not use these simplifications to make the model applicable in the case of a moderately compliant thin wedge.

Intermediate rigidity $\left(1 \gtrsim \frac{\mu_0}{\mu_*} \gtrsim \frac{1}{\Theta_0^2}\right)$. In this case, approximations (15) and (18) are applicable, as well as Eqs. (16) and (19). In view of the last two, it is reasonable to include u_ϑ^A and p_r^A in a new set of four variables to be expressed via $u_\vartheta^A,\ p_r^A$ and some two other variables. From (15) and (18), it appears that the two remaining variables to be included into the set are those, which do not tend to infinity when Θ_0 tends to zero. Such are u_ϑ^S in (15) and p_r^S in (18). Finally, the set contains variables $u_\vartheta^S,\ p_r^S,\ u_\vartheta^A$ and p_r^A; they are to be expressed via the remaining variables $u_r^S,\ p_\vartheta^S,\ u_r^A$ and p_ϑ^A.

For the external system, the Eq. (5) yield

$$\begin{aligned}
u_\vartheta^S &= c_{11}u_r^S + c_{12}p_\vartheta^S + c_{13}u_r^A + c_{14}p_\vartheta^A,\\
p_r^S &= c_{21}u_r^S + c_{22}p_\vartheta^S + c_{23}u_r^A + c_{24}p_\vartheta^A,\\
u_\vartheta^A &= c_{31}u_r^S + c_{32}p_\vartheta^S + c_{33}u_r^A + c_{34}p_\vartheta^A,\\
p_r^A &= c_{41}u_r^S + c_{42}p_\vartheta^S + c_{43}u_r^A + c_{44}p_\vartheta^A,
\end{aligned} \tag{26}$$

where c_{kj} are known coefficients. Note that the dimensionless coefficients c_{11}, c_{13}, c_{22}, c_{24}, c_{31}, c_{33}, c_{42}, c_{44}, being independent on Θ_0, may be assumed to have order $O(1)$ in $\Theta_0(k = 1,\dots,4)$. Similarly, the dimensionless coefficients $\frac{1}{\mu_*}c_{12},\ \frac{1}{\mu_*}c_{14},\ \mu_*c_{21},\ \mu_*c_{23},\ \frac{1}{\mu_*}c_{32},\ \frac{1}{\mu_*}c_{34},\ \mu_*c_{41},\ \mu_*c_{44}$ are also of order $O(1)$ in Θ_0. Then substitution of (16) and (19) into (26) and comparing orders of terms implies that the sums $c_{k3}u_r^A + c_{k4}p_\vartheta^A$ are of order $O(\Theta_0)$ as compared with the sums $c_{k1}u_r^S + c_{k2}p_\vartheta^S$; consequently, they may be neglected to this accuracy. In particular, the first two of Eq. (26) for the external system become

$$\begin{aligned}
u_\vartheta^S &= c_{11}u_r^S + c_{12}p_\vartheta^S,\\
p_r^S &= c_{21}u_r^S + c_{22}p_\vartheta^S.
\end{aligned} \tag{27}$$

For the thin wedge, we resolve the symmetric parts of (15) (or, equivalently, of (18)) with respect to the same variables u_ϑ^S and p_r^S. The result is

$$\begin{aligned}
u_\vartheta^S &= e_{11}\Theta_0u_r^S + e_{12}\Theta_0\frac{1}{\mu^0}p_\vartheta^S,\\
p_r^S &= e_{21}\Theta_0\mu_0u_r^S + e_{22}\Theta_0p_\vartheta^S,
\end{aligned} \tag{28}$$

where

$$e_{11} = -\frac{(k_0-1)s + (2k_0-1)}{2k_0},\quad e_{12} = \frac{2k_0-1}{4k_0},$$
$$e_{21} = -\frac{(s+1)^2}{k_0},\quad e_{22} = \frac{(s+1) - k_os}{2k_0}.$$

Joining (27) and (28) yields

$$u_\vartheta^S : c_{11}u_r^S + \frac{1}{\mu_*}(\mu_*c_{12})p_\vartheta^S = e_{11}\Theta_0u_r^S + e_{12}\Theta_0\frac{1}{\mu_0}p_\vartheta^S,$$
$$p_r^S : \mu_*\left(\frac{1}{\mu_*}c_{21}\right)u_r^S + c_{22}p_\vartheta^S = e_{21}\Theta_0\mu_0u_r^S + e_{22}\Theta_0p_\vartheta^S.$$

For sufficiently small Θ_0, by neglecting $e_{11}\Theta_0$ as compared with c_{11} and $e_{22}\Theta_0$ as compared with c_{22}, we may write these equations as:

$$\begin{aligned}
u_\vartheta^S &: c_{11}u_r^S + \frac{1}{\mu_*}(\mu_*c_{12})p_\vartheta^S = e_{12}\Theta_0\frac{1}{\mu_0}p_\vartheta^S,\\
p_r^S &: \mu_*\left(\frac{1}{\mu_*}c_{21}\right)u_r^S + c_{22}p_\vartheta^S = e_{21}\Theta_0\mu_0u_r^S.
\end{aligned} \tag{29}$$

The homogeneous system (29) provides approximate values of roots of $\Delta(s)$ for an arbitrary ratio $\frac{\mu_0}{\mu_*}$ in the considered range to the accuracy $O(\Theta_0)$. In particular cases, we have further simplifications.

Compliant or moderately rigid thin wedge $\left(\frac{\mu_0}{\mu_*}\Theta_0 \ll 1\right)$. Then in the second equation of (29) its r.h.s. is much less than the first term in the l.h.s. This yields

$$p_r^S : \mu_*\left(\frac{1}{\mu_*}c_{21}\right)u_r^S + c_{22}p_\vartheta^S = 0,$$

what corresponds to the continuous shear traction across the thin wedge. Besides, substitution of (16) into the third of (18) and comparison with the third row provided by (9), shows that p_ϑ^A may be assumed zero even to the higher accuracy of $\frac{\mu_0}{\mu_*}\Theta_0^3$ as compared with the unit. Therefore, the normal tractions may be also considered continuous ($p_\vartheta^A = 0$). We see that the traction continuity conditions (20) are acceptable in the considered case. Two remaining equations in (18) provide two additional conditions (21). Hence, the model 1 is applicable in a rather wide range of rigidities; it is certainly of use when $\frac{\mu_0}{\mu_*}\Theta_0 \ll 1$.

Rigid or moderately compliant thin wedge ($\frac{\mu_*}{\mu_0}\Theta_0 \ll 1$). Then in the first equation of (29) its r.h.s. is much less than the second term in the l.h.s. This yields

$$u_\vartheta^S : c_{11}u_r^S + \frac{1}{\mu_*}(\mu_*c_{12})p_\vartheta^S = 0,$$

what corresponds to the continuous normal displacement across the thin wedge. Besides, substitution of (19) into the fourth of (15) and comparison with the fourth row provided by (5), shows that u_r^A may be assumed zero to the accuracy of $\frac{\mu_0}{\mu_*}\Theta_0$ as compared with the unit. Therefore, the shear displacements may be also considered continuous ($u_r^A = 0$). We see that the displacement continuity conditions (23) are acceptable in the considered case. Two remaining equations in (15) provide two additional conditions (24). Hence, the model 2 is applicable in a rather wide range of rigidities; it is certainly of use when $\frac{\mu_*}{\mu_0}\Theta_0 \ll 1$.

Thin wedge is either moderately compliant, or moderately rigid, or its rigidity is similar to that of neighbor wedges. This is the case when $\Theta_0 \ll \frac{\mu_0}{\mu_*} \ll 1/\Theta_0$. Then both pairs (20) and (23) of the continuity conditions are applicable. All the components of displacements and tractions are continuous across the thin wedge:

$$u_\vartheta^S = 0,\quad u_r^A = 0,\quad p_r^S = 0,\quad p_\vartheta^A = 0. \tag{30}$$

This is the case of so-called "ideal" contact. Four Eq. (30) present the simplest physical model, which is applicable when $\Theta_0 \ll \frac{\mu_0}{\mu_*} \ll 1/\Theta_0$. We shall call it the *model 3*. In particular, it may be expected that for the angle $\Theta_0 = 0.5^0$, the model 3 is applicable in the interval of rigidities $0.1 \leqslant \frac{\mu_0}{\mu_*} \leqslant 10$.

The corresponding approximate system is

$$\begin{aligned}
u_\vartheta^S &: a_{11}p_\vartheta^S + a_{14}p_r^A = 0,\\
u_r^A &: a_{41}p_\vartheta^S + a_{44}p_r^A = 0.
\end{aligned} \tag{31}$$

Note that in the analysis presented the possibility to neglect a small angle Θ_0 appears in two distinct instances: (i) in the geometrical scheme including a thin wedge, and (ii) in the contact conditions simulating the influence of the wedge. The models 1 and 2 cover the cases when the influence of the thin wedge with a small angle Θ_0 cannot be neglected in the contact conditions (21) and (24), respectively. In contrast, the model 3 within the range of its applicability assumes that $\Theta_0 = 0$ in the contact conditions, while the angle Θ_0 may be not small in a geometrical scheme. Only when it is small and the model 3 is applicable, we may set $\Theta_0 = 0$ both in the geometrical scheme and in the contact conditions. Then the results coincide with those for a system of $m - 1$ wedges having ideal

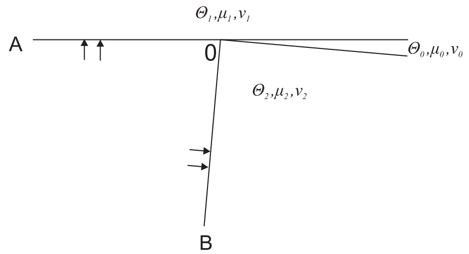

Fig. 3. An open system of three wedges.

contact between each pair of wedges. Henceforth, we shall address this case as that, for which the thin wedge is excluded from consideration.

6. Numerical results

Consider an open system (Fig. 3) of three wedges containing a thin wedge with the angle Θ_0, which is taken 5°, 1° and 0.1°. With the purpose to provide benchmarks (Helsing and Jonsson, 2002) for comparison with data, which may be obtained by other methods, we present numerical results in table with eight correct significant digits. To make the comprehension of the data easier, they are also presented diagrammatically in Figs. 4 and 5.

Two external wedges have angles $\Theta_1 = 180°$ and $\Theta_2 = 95° - \Theta_0$, thus the angle of the notch *AOB* between external wedges is constant and equals 85°. The ratio of the shear modules of external wedges is $\frac{\mu_2}{\mu_1} = 2$. The shear modulus μ_0 of the thin wedge is taken various to simulate cases of very high, very low and intermediate rigidity: $\mu = \frac{\mu_0}{\mu_1} = 10^3, 10^2, 10, 1, 10^{-1}, 10^{-2}$. The Poisson's ratio is the same for all the wedges: $\nu_1 = \nu_2 = \nu_0 = 0.3$.

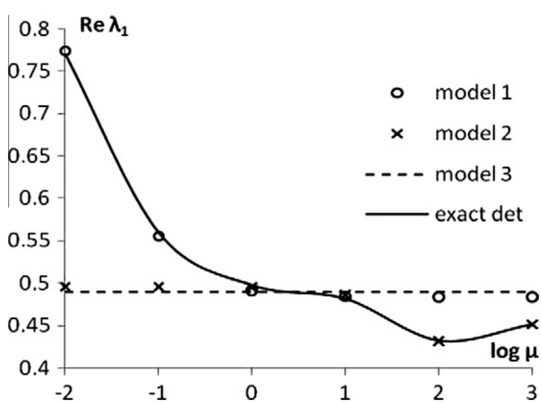

Fig. 4. Dependence Re λ_1 on the ratio of rigidities $\mu = \mu_0/\mu_1$ for $\Theta_0 = 5°$.

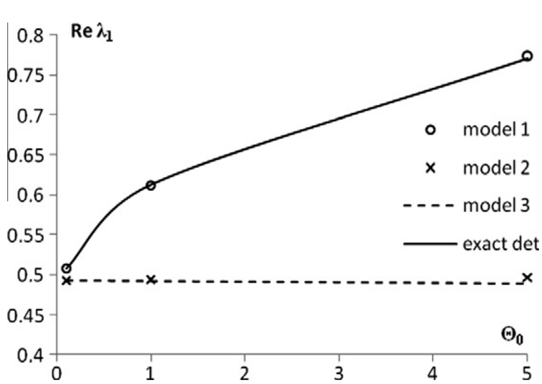

Fig. 5. Dependence Re λ_1 on the angle Θ_0 for a compliant wedge ($\mu = 0.01$).

It can be seen that the numerical results are in complete agreement with the theoretical analysis of the previous section. The Table 1 gives also quantitative data on the ranges of applicability and actual accuracy of the models 1, 2 and 3, Specifically,

1. In the case of the a rigid thin wedge $\left(\frac{\mu_0}{\mu_1} \geqslant 10\right)$, the model 2 provides the accuracy of two significant digits even for $\Theta_0 = 5°$. With decreasing angle Θ_0, the accuracy and the range of the model applicability increase: for $\Theta_0 = 0.1°$, the error does not exceed one unit in the fourth significant digit when $\frac{\mu_0}{\mu_1} \geqslant 1$.
   In contrast, the model 1 does not provide acceptable results for a rigid thin wedge $\left(\frac{\mu_0}{\mu_1} \geqslant 10^2\right)$ when $\Theta_0 \geqslant 1°$. However, we may see that for a fixed high rigidity, the model 1 becomes acceptable when the angle Θ_0 is sufficiently small to guarantee the applicability of the model 3.
2. In the case of very compliant thin wedge ($\frac{\mu_0}{\mu_1} \leqslant 10^{-2}$), the model 1 provides the accuracy of two significant digits even for $\Theta_0 = 5°$. With decreasing angle Θ_0, the accuracy and the range of the model applicability increase: for $\Theta_0 = 1°$, the accuracy of two significant digits is observed up to $\frac{\mu_0}{\mu_1} = 1$.
   In contrast, the model 2 is unacceptable for a compliant thin wedge ($\frac{\mu_0}{\mu_0} \leqslant 10^{-1}$) when $\Theta_0 \geqslant 1°$. However, we may see that for a fixed low rigidity, the model 2 becomes acceptable when the angle Θ_0 is sufficiently small to guarantee the applicability of the model 3.

Table 1
Values of $\lambda = 2 + s^*(\mu = \mu_0/\mu_1)$.

μ	*exact det*	*model* 1	*model* 2	*model* 3
$\Theta_0 = 5^0$				
10^3	λ_1 = .45135676	λ_1 = .48345091	λ_1 = .45088799	λ_1 = .48875019
	± .07534376*i*	λ_2 = .16709769	± .07261039*i*	λ_2 = .16862378
10^2	λ_1 = .43189228	λ_1 = .48352386	λ_1 = .43174654	λ_1 = .48875019
	± .05068506*i*	λ_2 = .16715282	± .04419697*i*	λ_2 = .16862378
10	λ_1 = .48142991	λ_1 = .48425262	λ_1 = .48629933	λ_1 = .48875019
	λ_2 = .26001808	λ_2 = .16770508	λ_2 = .25781700	λ_2 = .16862378
1	λ_1 = .49738090	λ_1 = .49146122	λ_1 = .49499949	λ_1 = .48875019
	λ_2 = .19577876	λ_2 = .17331591	λ_2 = .19102658	λ_2 = .16862378
10^{-1}	λ_1 = .55963467	λ_1 = .55496591	λ_1 = .49593627	λ_1 = .48875019
	λ_2 = .24945793	λ_2 = .23543348	λ_2 = .18220732	λ_2 = .16862378
10^{-2}	λ_1 = .77058379	λ_1 = .77345853	λ_1 = .49603078	λ_1 = .48875019
	λ_2 = .57705838	λ_2 = .57067148	λ_2 = .18129787	λ_2 = .16862378
$\Theta_0 = 1^0$				
10^3	λ_1 = .44501544	λ_1 = .49072553	λ_1 = .44496920	λ_1 = .49180431
	± .06982598*i*	λ_2 = .19545088	± .06914209*i*	λ_2 = .19577514
10^2	λ_1 = .47072288	λ_1 = .49073951	λ_1 = .47202017	λ_1 = .49180431
	λ_2 = .31880070	λ_2 = .19546432	λ_2 = .31797147	λ_2 = .19577514
10	λ_1 = .49004835	λ_1 = .49087934	λ_1 = .49098607	λ_1 = .49180431
	λ_2 = .21599496	λ_2 = .19559873	λ_2 = .21602159	λ_2 = .19577514
1	λ_1 = .49348296	λ_1 = .49227494	λ_1 = .49302535	λ_1 = .49180431
	λ_2 = .20111367	λ_2 = .19694603	λ_2 = .19994635	λ_2 = .19577514
10^{-1}	λ_1 = .50727047	λ_1 = .50595023	λ_1 = .49323196	λ_1 = .49180431
	λ_2 = .21320576	λ_2 = .21070175	λ_2 = .19824646	λ_2 = .19577514
10^{-2}	λ_1 = .61265689	λ_1 = .61204077	λ_1 = .49325265	λ_1 = .49180431
	λ_2 = .34657307	λ_2 = .34440460	λ_2 = .19807550	λ_2 = .19577514
$\Theta_0 = 0.1^0$				
10^3	λ_1 = .47054905	λ_1 = .49230888	λ_1 = .47068184	λ_1 = .49241718
	λ_2 = .32110942	λ_2 = .20157343	λ_2 = .32102605	λ_2 = .20160616
10^2	λ_1 = .49014107	λ_1 = .49231027	λ_1 = .49024906	λ_1 = .49241718
	λ_2 = .21952656	λ_2 = .20157482	λ_2 = .21954071	λ_2 = .20160616
10	λ_1 = .49223501	λ_1 = .49232412	λ_1 = .49232818	λ_1 = .49241718
	λ_2 = .20367472	λ_2 = .20158877	λ_2 = .20369055	λ_2 = .20160616
1	λ_1 = .49258381	λ_1 = .49246226	λ_1 = .49253862	λ_1 = .49241718
	λ_2 = .20213847	λ_2 = .20172832	λ_2 = .20201643	λ_2 = .20160616
10^{-1}	λ_1 = .49398596	λ_1 = .49384494	λ_1 = .49255969	λ_1 = .49241718
	λ_2 = .20336831	λ_2 = .20312685	λ_2 = .20184808	λ_2 = .20160616
10^{-2}	λ_1 = .50752393	λ_1 = .50739650	λ_1 = .49256180	λ_1 = .49241718
	λ_2 = .21761392	λ_2 = .21737340	λ_2 = .20183124	λ_2 = .20160616

3. In accordance with the theory, the model 3 of ideal contact is applicable for an arbitrary rigidity of the thin wedge if its angle Θ_0 is very small. The data of Table 1 show that if the angle $\Theta_0 = 0.1°$, the error of this model is less then two units in the second significant digit in the wide range of the ratio of rigidities $10^{-2} \leqslant \frac{\mu_0}{\mu_1} \leqslant 10^2$. With growing angle Θ_0, the accuracy and the range of the model applicability decrease: for $\Theta_0 = 1°$, the range becomes $10^{-1} \leqslant \frac{\mu_0}{\mu_1} < 10$; for $\Theta_0 = 5°$, the model 3 is inapplicable for any ratio $\frac{\mu_0}{\mu_1}$.

The Table 1 confirms also that in the range of the model 3 applicability, both the models 1 and 2 are applicable, as well.

The case, when the angle Θ_0 of the thin wedge becomes very small, deserves special comments. Note that the denominators in the exact Eq. (13) tend to zero when $\Theta_0 \to 0$. This unavoidably leads to deterioration of numerical results for the exact system if not using asymptotic expansions in Θ_0. Thus one may expect growing instability of the exact solution when the angle Θ_0 becomes too small. Numerical experiments with decreasing Θ_0 give an idea when it happens. We could see that for $\Theta_0 = 0.1°$, the figure in the ninth digit after the decimal point becomes unstable: it changes when taking different starting points of Muller's iterations. With further decrease of Θ_0 to 0.01° deterioration occurred in the forth digit. For $\Theta_0 = 0.001°$ the results were absolutely wrong.

Meanwhile, the model 3 provides stable and accurate results for whatever small values of Θ_0. This implies that for very small values of the angle Θ_0, exclusion of the thin wedge from consideration is superior over accounting for it in numerical calculations. As a benchmark, we may recommend to exclude the thin wedge when $\Theta_0 \leqslant 0.1°$ in a quite wide range of the ratio of rigidities $10^{-2} \leqslant \frac{\mu_0}{\mu_1} \leqslant 10^2$. The range extends to infinity when $\Theta_0 \to 0$.

Similar results are obtained for other configurations.

7. Conclusions

The conclusions of the work are as follows.

1. The theoretical analysis discloses that with decreasing angle Θ_0 of a thin wedge, the asymptotic equations for the wedge become linearly dependent what leads to deterioration of solution when employing the approximate system (14) or (17). The very difficulty suggests a key to overcome it by using Eqs. (16) and (19), which follow from the linear dependence. This yields the model 1 of continuous tractions and the model 2 of continuous displacements in the cases of, respectively, compliant or moderately rigid thin wedge $\left(\frac{\mu_0}{\mu_*}\Theta_0 \ll 1\right)$ and rigid or moderately compliant thin wedge $\left(\frac{\mu_*}{\mu_0}\Theta_0 \ll 1\right)$. The model 3 of ideal contact becomes available when $\Theta_0 \ll \frac{\mu_0}{\mu_*} \ll \frac{1}{\Theta_0}$.
2. Numerical experiments show that the model 1 of continuous tractions provides quite accurate results (at least two correct significant digits) even for the angle $\Theta_0 = 5°$ when $\frac{\mu_0}{\mu_*} \leqslant 10^{-2}$. The model 2 of continuous displacements provides this accuracy for the same angle when $\frac{\mu_0}{\mu_*} \geqslant 10$. With decreasing angle Θ_0, the accuracy provided by the models increases and the ranges of their applicability start to overlap. The interval of overlapping corresponds to the range where the model 3 of ideal contact is applicable. For $\Theta_0 = 0.1°$, the range of its applicability is $10^{-2} \leqslant \frac{\mu_0}{\mu_*} \leqslant 10^2$. In this case, the thin wedge may be excluded from consideration.
3. For very small values of the angle $\Theta_0(\Theta_0 < 0.01°)$, exclusion of the thin wedge from consideration is superior over accounting for it as concerning with the efficiency, accuracy and stability of calculations. Specifically, it is safe to exclude the thin wedge in the wide range $\frac{1}{A} \leqslant \frac{\mu_0}{\mu_*} \leqslant A$ with $A = 10^2$ for $\Theta_0 = 0.1°$ ($A \to \infty$ when $\Theta_0 \to 0$).

Acknowledgements

The work is supported by the Polish Ministry of Science and Higher Education as a scientific project for 2010–2012 (N N519 440739). The authors also appreciate the support of the Russian Fund of Fundamental Researches (09-05-00104) at the early stage of the work.

References

Blinova, V., Linkov, A., 1995. A method of finding asymptotic forms at the common apex of elastic wedges. J. Appl. Math. Mech. 59, 187–195. Trans. Prikl. Mat. Mekh. (1995), 59: 199–208.

Churchman, C., Mugadu, A., Hills, D.A., 2004. Asymptotic results for slipping complete frictional contacts. Eur. J. Mech. A-Solid. 22, 793–800.

Comninou, M., 1976. Stress singularity at a sharp edge in contact problems with friction. J. Appl. Math. Phys. (ZAMP) 27, 493–499.

Dobroskok, A.A., Linkov, A.M., 2010. CV BEM for transient poro- (thermo-) elastic problems concerning with blocky systems with singular points and lines of discontinuities. Int. J. Eng. Sci. 48, 658–669.

Dundurs, J., Lee, M.-S., 1972. Stress concentration at a sharp edge in contact problems. J. Elasticity 2, 109–112.

Gdoutos, E.E., Theocaris, P.S., 1975. Stress concentrations at the apex of a plane indenter acting on an elastic half plane. J. Appl. Mech. 42, 688–692.

Helsing, J., Jonsson, A., 2002. On the accuracy of benchmarks tables and graphical results in the applied mechanics literature. ASME J. Appl. Mech. 69, 88–90.

Linkov, A., Koshelev, V., 2005. Multi-wedge points and multi-wedge elements in computational mechanics: evaluation of exponent and angular distribution. Int. J. Solids Struct. 43, 5909–5930.

Linkov, A., Rybarska-Rusinek, L., 2008. Numerical methods and models for anti-plane strain of a system with a thin elastic wedge. Arch. Appl. Mech. 78, 821–831.

Linkov, A., Rybarska-Rusinek, L., 2010. Interface conditions simulating influence of a thin elastic wedge with smooth contacts. Arch. Appl. Mech. (accepted for publication).

Mishuris, G.S., 1997. 2-D boundary value problems of thermoelasticity in a multi-wedge-multi-layered region. Part 1. Sweep method. Arch. Mech. 49, 1103–1134.

Mishuris, G., Kuhn, G., 2001. Comparative study of an interface crack for different wedge-interface models. Arch. Appl. Mech. 71, 764–780.

Mugadu, A., Hills, D.A., Barber, J.R., Sackfield, A., 2004. The application of asymptotic solutions to characterizing the process zone in almost complete frictional contacts. Int. J. Solids Struct. 41, 385–397.

Paggi, M., Carpinteri, A., 2008. On the Stress Singularities at Multimaterial Interfaces and Related Analogies With Fluid Dynamics and Diffusion. Appl. Mech. Rev. 61, 020801.

Sinclair, G.B., 2004a. Stress singularities in classical elasticity - I: Removal, interpretation, and analysis. Appl. Mech. Rev. 57 (4), 251–297.

Sinclair, G.B., 2004b. Stress singularities in classical elasticity - II: Asymptotic identification. Appl. Mech. Rev. 57 (5), 385–439.